Analysis of Random Block Design with one missing observations

Let the observation $y_{ij} = x$, say, in the j^{th} block and receiving the i^{th} treatment be missing, as given below.

Blocks	Treatments									Totals
	1	2	...	i	...	v				
1	y_{11}	y_{21}	...	y_{i1}	...	y_{v1}				$T'_{.1}$
2	y_{12}	y_{22}	...	y_{i2}	...	y_{v2}				$T'_{.2}$
⋮	⋮									
j	y_{1j}	y_{2j}	...	$\boxed{y_{ij} = x}$	...	y_{vj}				$T'_{.j} + x$
⋮	⋮	⋮		⋮		⋮				⋮
r	y_{1r}	y_{2r}	...	y_{ir}	...	y_{vr}				$T'_{.r}$
Totals	$T'_{1.}$	$T'_{2.}$	...	$T_{i.} + x$		$T'_{v.}$				$T_{..} + x = G$

where

$T'_{i.}$ is the total of observations for the treatments not containing the missing observations
$(i' = 1, 2, \ldots, i-1, i+1, \ldots, v)$.

$T'_{.j}$ is the total of observations in the blocks not containing the missing observations
$(j' = 1, 2, \ldots, j-1, j+1, \ldots, r)$.

$T_{i.}$ is the total of $(r-1)$ known observations for the i^{th} treatment (containing the missing observations)

$T_{.j}$ is the total of $(v-1)$ known observation in the j^{th} block (containing the missing observation)

$T_{..}$ is the total of all $(vr-1)$ known observations.

The linear model is

$$y_{ij} = \mu + t_i + b_j + \epsilon_{ij} \; ; \quad \begin{array}{l} i = 1, 2, \ldots, v \\ j = 1, 2, \ldots, r \end{array} \quad \rightarrow (1)$$

Split this linear model in two parts - one corresponding to the known observations and the other corresponding to the unknown (missing) observation as given below:

$$y_{i'j'} = \mu + t_{i'} + b_{j'} + \epsilon_{i'j'} \; ; \quad \begin{pmatrix} i' = 1, 2, \ldots, i-1, i+1, \ldots, v \\ j' = 1, 2, \ldots, j-1, j+1, \ldots, r \end{pmatrix} \quad \rightarrow (2)$$

and $$y_{ij} = x = \mu + t_i + b_j + \epsilon_{ij} \rightarrow (3)$$

where x denotes the missing observations corresponding to the i^{th} treatment in the j^{th} block.

The least square estimates of the parameters in (2) or (3) are given by

$$\hat{\mu} = \bar{y}_{..} = \frac{T_{..} + \hat{x}}{vr} \rightarrow (4)$$

$$\hat{t}_i' = \bar{y}_{i.}' - \bar{y}_{..} = \frac{T_{i.}'}{r} - \frac{T_{..} + \hat{x}}{vr} \rightarrow (5)$$

$$\hat{b}_j' = y_{.j}' - \bar{y}_{..} = \frac{T_{.j}'}{v} - \frac{T_{..} + \hat{x}}{vr} \rightarrow (6)$$

$$\hat{t}_i = \bar{y}_{i.} - \bar{y}_{..} = \frac{T_{i.} + \hat{x}}{r} - \frac{T_{..} + \hat{x}}{vr} \rightarrow (7)$$

$$\hat{b}_j = \bar{y}_{.j} - \bar{y}_{..} = \frac{T_{.j} + \hat{x}}{v} - \frac{T_{..} + \hat{x}}{vr} \rightarrow (8)$$

From (3), $\hat{x} = \hat{y}_{ij} = \hat{\mu} + \hat{t}_i + \hat{b}_j + \hat{\epsilon}_{ij}$

Since there is only one missing observations, we may take $\hat{\epsilon}_{ij} = 0$.

$$\therefore \hat{x} = \hat{\mu} + \hat{t}_i + \hat{b}_j$$

$$= \frac{T_{..} + \hat{x}}{vr} + \frac{T_{i.} + \hat{x}}{r} - \frac{T_{..} + \hat{x}}{vr} + \frac{T_{.j} + \hat{x}}{v} - \frac{T_{..} + \hat{x}}{vr}$$

(using (4), (7), (8))

$$\Rightarrow \hat{x} = \frac{T_{i.} + \hat{x}}{r} + \frac{T_{.j} + \hat{x}}{v} - \frac{T_{..} + \hat{x}}{vr}$$

$$\Rightarrow \hat{x}\left(1 - \frac{1}{r} - \frac{1}{v} + \frac{1}{vr}\right) = \frac{T_{i.}}{r} + \frac{T_{.j}}{v} - \frac{T_{..}}{vr}$$

$$\Rightarrow \hat{x}(vr - v - r + 1) = v(T_{i.}) + r\cdot(T_{.j}) - T_{..}$$

$$\hat{x}[v(r-1) - (r-1)] = v.T_{i.} + rT_{.j} - T_{..}$$

$$\hat{x}[(r-1)(v-1)]$$

$$\therefore \boxed{\hat{x} = \frac{vT_{i.} + rT_{.j} - T_{..}}{(r-1)(v-1)}}$$